Linear Algebra, Spring 2005

Solutions

May 4, 2005

Problem 8.38

(a) $\begin{vmatrix} 2 & 6 \\ 4 & 1 \end{vmatrix} = (2)(1) - (6)(4) = 2 - 24 = -22$

(c) $\begin{vmatrix} -2 & 8 \\ -5 & -3 \end{vmatrix} = \{(-2)(-3)\} - \{8(-5)\} = 6 - (-40) = 46$

(e) $\begin{vmatrix} a-b & a \\ b & a+b \end{vmatrix} = \{(a-b)(a+b)\} - \{(a)(b)\} = (a^2 - b^2) - (ab) = a^2 - ab - b^2$

Problem 8.40

(a) $\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{vmatrix}$

$= 2(5)(4) + 1(-2)(1) + 1(-3)(0) - 1(5)(1) - (-3)(-2)(2) - 4(1)(0)$

$= 40 - 2 + 0 - 5 - 12 - 0 = 21$

(b) $\begin{vmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{vmatrix}$

$$= 3(5)(1) + (-2)(-1)(0) + (-4)(6)(2) - 0(5)(-4) - 6(-1)(3) - 1(-2)(2)$$
$$= 15 + 0 - 48 - 0 + 18 + 4 = -11$$

Alternate Solution

(a) $$\begin{vmatrix} 2 & 1 & 1 \\ 0 & 5 & -2 \\ 1 & -3 & 4 \end{vmatrix} = 2\begin{vmatrix} 5 & -2 \\ -3 & 4 \end{vmatrix} - 1\begin{vmatrix} 0 & -2 \\ 1 & 4 \end{vmatrix} + 1\begin{vmatrix} 0 & 5 \\ 1 & -3 \end{vmatrix}$$

$$= 2(20 - 6) - (0 + 2) + (0 - 5) = 28 - 2 - 5 = 21$$

(b) $$\begin{vmatrix} 3 & -2 & -4 \\ 2 & 5 & -1 \\ 0 & 6 & 1 \end{vmatrix} = 3\begin{vmatrix} 5 & -1 \\ 6 & 1 \end{vmatrix} - (-2)\begin{vmatrix} 2 & -1 \\ 0 & 1 \end{vmatrix} + (-4)\begin{vmatrix} 2 & 5 \\ 0 & 6 \end{vmatrix}$$

$$= 3(5 + 6) + (2)(2 - 0) + (-4)(12 - 0) = 33 + 4 - 48 = -11$$

Note: For both parts you can also expand by first column as these have one zero entry.

Problem 8.41

(a)

$$\begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix}$$

Since there are only two non-zero elements in the second row, apply the column operations:

$$\begin{vmatrix} 1 & 2 & 2 & 3 \\ 1 & 0 & -2 & 0 \\ 3 & -1 & 1 & -2 \\ 4 & -3 & 0 & 2 \end{vmatrix} = \begin{vmatrix} 1 & 2 & 4 & 3 \\ 1 & 0 & 0 & 0 \\ 3 & -1 & 7 & -2 \\ 4 & -3 & 8 & 2 \end{vmatrix} = -(1)\begin{vmatrix} 2 & 4 & 3 \\ -1 & 7 & -2 \\ -3 & 8 & 2 \end{vmatrix} = \begin{vmatrix} 2 & 4 & 3 \\ 1 & -7 & 2 \\ -3 & 8 & 2 \end{vmatrix}$$

$= 2(-7)(2) + 4(2)(-3) + 3(8)(1) - (-3)(-7)(3) - 8(2)(2) - 2(4)(1)$

$= -28 - 24 + 24 - 63 - 32 - 8 = -131$

(b)

$$\begin{vmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{vmatrix}$$

Use row operations $R_3 = R_3 + R_1$ and $R_4 = R_4 - 2R_1$:

$$\begin{vmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 1 & -1 & 4 & 3 \\ 2 & 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 2 & 1 & 3 & 2 \\ 3 & 0 & 1 & -2 \\ 3 & 0 & 7 & 5 \\ -2 & 0 & -7 & -3 \end{vmatrix} = -(1)\begin{vmatrix} 3 & 1 & -2 \\ 3 & 7 & 5 \\ -2 & -7 & -3 \end{vmatrix} = \begin{vmatrix} 3 & 1 & -2 \\ -3 & -7 & -5 \\ -2 & -7 & -3 \end{vmatrix}$$

Use row operation $R_2 = R_2 + R_1$:

$$\begin{vmatrix} 3 & 1 & -2 \\ 0 & -6 & -7 \\ -2 & -7 & -3 \end{vmatrix} = 3\begin{vmatrix} -6 & -7 \\ -7 & -3 \end{vmatrix} - 0\begin{vmatrix} 1 & -2 \\ -7 & -3 \end{vmatrix} + (-2)\begin{vmatrix} 1 & -2 \\ -6 & -7 \end{vmatrix}$$

$= 3(18 - 49) - 0 - 2(-7 - 12) = -93 + 38 = -55$

Problem 8.43(a)

$$\begin{vmatrix} 1 & 2 & -1 & 3 & 1 \\ 2 & -1 & 1 & -2 & 3 \\ 3 & 1 & 0 & 2 & -1 \\ 5 & 1 & 2 & -3 & 4 \\ -2 & 3 & -1 & 1 & -2 \end{vmatrix}$$

Use row operations $R_1 = R_1 + R_2$, $R_4 = R_4 - 2R_2$, $R_5 = R_5 + R_2$

$$\begin{vmatrix} 3 & 1 & 0 & 1 & 4 \\ 2 & -1 & 1 & -2 & 3 \\ 3 & 1 & 0 & 2 & -1 \\ 1 & 3 & 0 & 1 & -2 \\ 0 & 2 & 0 & -1 & 1 \end{vmatrix} = -(1) \begin{vmatrix} 3 & 1 & 1 & 4 \\ 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & -2 \\ 0 & 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} -3 & -1 & -1 & -4 \\ 3 & 1 & 2 & -1 \\ 1 & 3 & 1 & -2 \\ 0 & 2 & -1 & 1 \end{vmatrix}$$

Use row operations $R_1 = R_1 + R_2$, $R_2 = R_2 - 3R_3$

$$\begin{vmatrix} 0 & 0 & 1 & -5 \\ 0 & -8 & -1 & 5 \\ 1 & 3 & 1 & -2 \\ 0 & 2 & -1 & 1 \end{vmatrix} = (1) \begin{vmatrix} 0 & 1 & -5 \\ -8 & -1 & 5 \\ 2 & -1 & 1 \end{vmatrix} = \begin{vmatrix} 0 & 1 & -5 \\ -8 & -1 & 5 \\ 2 & -1 & 1 \end{vmatrix}$$

$$= 0 \begin{vmatrix} -1 & 5 \\ -1 & 1 \end{vmatrix} - (1) \begin{vmatrix} -8 & 5 \\ 2 & 1 \end{vmatrix} + (-5) \begin{vmatrix} -8 & -1 \\ 2 & -1 \end{vmatrix}$$

$$= 0 - (-8 - 10) - 5(8 + 2) = 18 - 50 = -32$$

Problem 8.44

(a)

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 2 & 1 \end{bmatrix}$$

$$det(A) = |A| = 1(1)(1) + 1(1)(0) + 0(2)(1) - 0(1)(0) - 2(1)(1) - 1(1)(1) = -2$$

$$A_{11} = + \begin{vmatrix} 1 & 1 \\ 2 & 1 \end{vmatrix} = -1, \quad A_{12} = - \begin{vmatrix} 1 & 1 \\ 0 & 1 \end{vmatrix} = -1, \quad A_{13} = + \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = 2$$

$$A_{21} = - \begin{vmatrix} 1 & 0 \\ 2 & 1 \end{vmatrix} = -1, \quad A_{22} = + \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix} = 1, \quad A_{23} = - \begin{vmatrix} 1 & 1 \\ 0 & 2 \end{vmatrix} = -2$$

$$A_{31} = + \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{32} = - \begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = -1, \quad A_{33} = + \begin{vmatrix} 1 & 1 \\ 1 & 1 \end{vmatrix} = 0$$

$$AdjA = \begin{bmatrix} -1 & -1 & 1 \\ -1 & 1 & -1 \\ 2 & -2 & 0 \end{bmatrix}$$

$$A^{-1} = \frac{AdjA}{|A|} = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} & -\frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} & \frac{1}{2} \\ -1 & 1 & 0 \end{bmatrix}$$

(b)

$$A = \begin{bmatrix} 1 & 2 & 2 \\ 3 & 1 & 0 \\ 1 & 1 & 1 \end{bmatrix}$$

$$det(A) = |A| = 1\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} - 2\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} + 2\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 1(1) - 2(3) + 2(2) = -1$$

$$A_{11} = +\begin{vmatrix} 1 & 0 \\ 1 & 1 \end{vmatrix} = 1, \quad A_{12} = -\begin{vmatrix} 3 & 0 \\ 1 & 1 \end{vmatrix} = -3, \quad A_{13} = +\begin{vmatrix} 3 & 1 \\ 1 & 1 \end{vmatrix} = 2$$

$$A_{21} = -\begin{vmatrix} 2 & 2 \\ 1 & 1 \end{vmatrix} = 0, \quad A_{22} = +\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = -1, \quad A_{23} = -\begin{vmatrix} 1 & 2 \\ 1 & 1 \end{vmatrix} = 1$$

$$A_{31} = +\begin{vmatrix} 2 & 2 \\ 1 & 0 \end{vmatrix} = -2, \quad A_{32} = -\begin{vmatrix} 1 & 2 \\ 3 & 0 \end{vmatrix} = 6, \quad A_{33} = +\begin{vmatrix} 1 & 2 \\ 3 & 1 \end{vmatrix} = -5$$

$$AdjA = \begin{bmatrix} 1 & 0 & -2 \\ -3 & -1 & 6 \\ 2 & 1 & -5 \end{bmatrix}$$

$$A^{-1} = \frac{AdjA}{|A|} = \begin{bmatrix} -1 & 0 & 2 \\ 3 & 1 & -6 \\ -2 & -1 & 5 \end{bmatrix}$$